Question 1

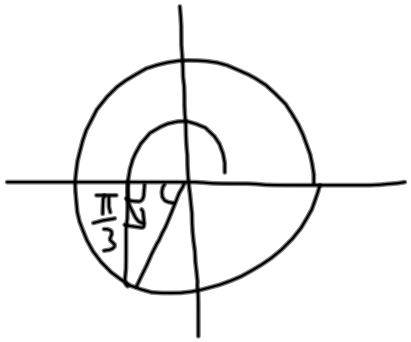
On the wrapping function, what are the coordinates of the point $P\left(\frac{-14\pi}{3}\right)$?

$-\frac{14\pi}{3} \div 2\pi = -\frac{14}{6}$

$= -\frac{7}{3} = -2\frac{1}{3}$

$-\frac{1}{3} \times 2\pi = -\frac{2\pi}{3} + 2\pi$

$t' = \frac{4\pi}{3}$



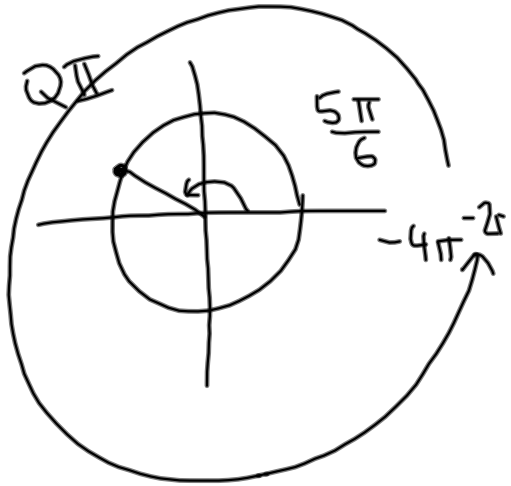
QIII → −, −

$P\left(-\frac{14\pi}{3}\right) = \left(-\cos\frac{\pi}{3}, -\sin\frac{\pi}{3}\right)$

$= \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Question 2

What angle t, in radians, corresponds to the coordinates $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$, where $t \in [-4\pi, -2\pi]$?



$$t = -4\pi + \frac{5\pi}{6}$$
$$= -\frac{19\pi}{6}$$

Question 3

If $f(x) = cot(x)$, find the value of $f\left(\frac{13\pi}{3}\right)$.

$\frac{13\pi}{3} \div 2\pi = \frac{13}{6}$

$= \frac{1}{6} \times 2\pi$

$= \frac{2\pi}{6} = \frac{\pi}{3}$

$t' = \frac{\pi}{3}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$f(x) = \cot x = \frac{\cos x}{\sin x}$

$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$f\left(\frac{13\pi}{3}\right) = \cot\frac{13\pi}{3} = \frac{\sqrt{3}}{3}$

Question 4

Given $f(x) = \cos x$ and $g(x) = \tan x$:

a) Find the values of x, in the interval $[-4\pi, 4\pi[$, where function f is at its minimum.

min of f: −1

cos x = −1 → Period = 2π

x = π, −3π, −π, 3π

x' = π + 2π = 3π

x'' = 3π + 2π = 5π

x''' = π − 2π = −π

x'''' = −π − 2π = −3π

x''''' = −3π − 2π = −5π

b) What are the increasing intervals of function g over the interval $\left]-\frac{3\pi}{2}, \frac{3\pi}{2}\right[$?

g(x) = tan x → Period = π

Asymptotes: $-\frac{\pi}{2}, \frac{\pi}{2}$

Always Increasing!

$\left]-\frac{3\pi}{2}, -\frac{\pi}{2}\right[\cup \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$

c) For which values of x, over the interval $[-2\pi, 2\pi[$, is $g(x) = 1$?

$g(x) = \tan x = 1 \quad \left(\frac{y}{x}\right)$

$x = \frac{\pi}{4}$ ✓ (since $x = y$)

Period $= \pi$

$x' = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$ ✓

$x'' = \frac{5\pi}{4} + \pi = \frac{9\pi}{4}$ ~~✗~~

$x''' = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ ✓

$x'''' = -\frac{3\pi}{4} - \pi = -\frac{7\pi}{4}$ ✓

$x''''' = -\frac{7\pi}{4} - \pi = -\frac{11\pi}{4}$ ~~✗~~

$X = \left\{-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}\right\}$

Question 5

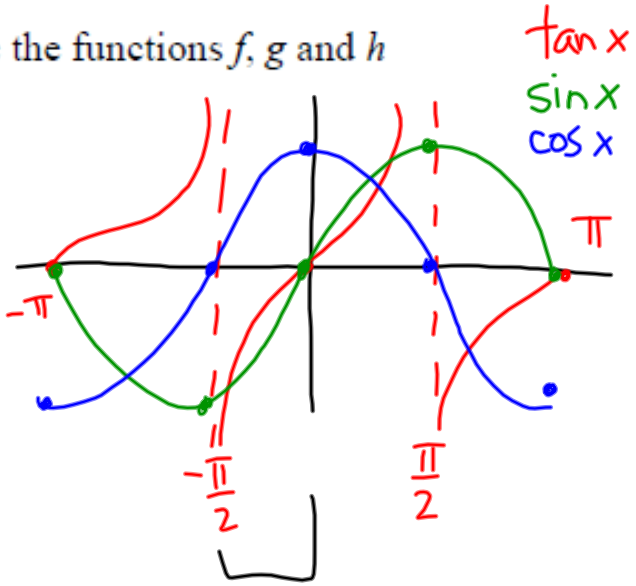
Given the functions $f(x) = \sin x$, $g(x) = \cos x$ and $h(x) = \tan x$:

a) Over the interval $[-\pi, \pi[$, determine the interval where f, g and h are simultaneously positive.

QI $\rightarrow$ $[0, \frac{\pi}{2}[$

b) Over the interval $[-\pi, \pi[$, for which values of x are the functions f, g and h simultaneously increasing?

$]-\frac{\pi}{2}, 0[$



c) If $x \in \Re$, for which values of x is $h(x)$ the minimum of f?

$h(x) = \tan x$

$f(x) = \sin x$

-1

$h(x) = -1$

$\tan x = -1 \quad \left(\frac{y}{x}\right)$

$x = \frac{3\pi}{4}$ (since $y = -x$)

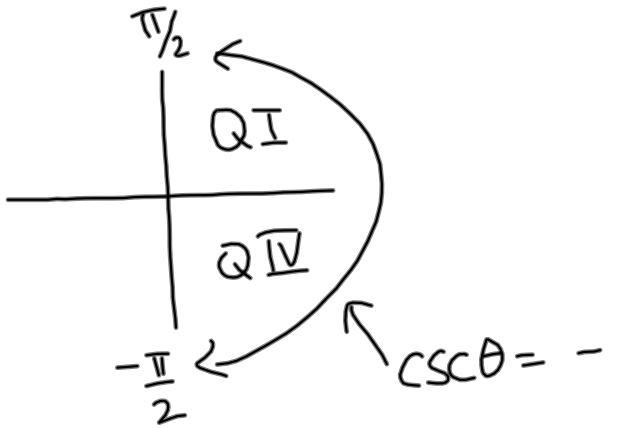
Period $= \pi$

$x = \frac{3\pi}{4} + k\pi$, where $k \in \mathbb{Z}$

Question 6

If $\csc\theta = -3$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, find the value of $\tan\theta$.

$\frac{1}{\sin} = \frac{hyp}{opp} = \frac{-3}{1}$



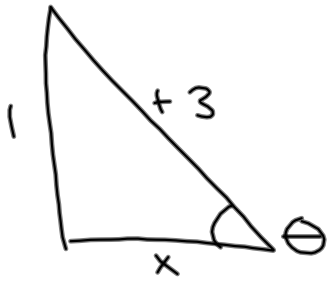
$\csc\theta = \frac{1}{\sin\theta}$

$\cot^2\theta + 1 = \csc^2\theta$

$\cot^2\theta = (-3)^2 - 1$

$\cot^2\theta = 8$

$\cot\theta = \pm\sqrt{8}$



$3^2 = 1^2 + x^2$

$x = \sqrt{8}$

$\tan\theta = \frac{opp}{adj} = \frac{1}{\sqrt{8}}$

$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{\pm\sqrt{8}} \rightarrow$ QIV $\tan\theta = -$

$\tan\theta = \frac{1}{-\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = -\frac{\sqrt{8}}{8} = -\frac{\sqrt{2}}{4}$

Question 7

Knowing that $x \in \Re$, solve the following equation:

$$4\sqrt{3} = 2\sin x + 5\sqrt{3}$$

$$4\sqrt{3} - 5\sqrt{3} = 2\sin x$$

$$-\frac{\sqrt{3}}{2} = \frac{2\sin x}{2}$$

$$\sin x = -\frac{\sqrt{3}}{2}$$

← y-value is $-\frac{\sqrt{3}}{2}$ on unit circle?

QIII $\frac{4\pi}{3}$, QIV $\frac{5\pi}{3}$

Period $= 2\pi$

$x' = \frac{4\pi}{3} + 2k\pi$, where $k \in \mathbb{Z}$

$x'' = \frac{5\pi}{3} + 2k\pi$, where $k \in \mathbb{Z}$

Question 8

Solve the following equation for $x \in [\pi, 3\pi]$:

$$2\sin^2 x + (-2+\sqrt{2})\sin x - \sqrt{2} = 0$$

$2\sin^2 x - 2\sin x + \sqrt{2}\sin x - \sqrt{2} = 0$

$2\sin x(\sin x - 1) + \sqrt{2}(\sin x - 1) = 0$

$(\sin x - 1)(2\sin x + \sqrt{2}) = 0$

$\sin x - 1 = 0$

$\sin x = 1$

$x = \frac{\pi}{2}$

Period $= 2\pi$

$x' = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$ ✓

No others

$2\sin x + \sqrt{2} = 0$

$\frac{2\sin x}{2} = -\frac{\sqrt{2}}{2}$

$\sin x = -\frac{\sqrt{2}}{2}$

$x' = \frac{5\pi}{4}$ ✓

$x'' = \frac{5\pi}{4} + 2\pi = \frac{9\pi}{4}$ ✓

No others

$x''' = \frac{7\pi}{4}$ ✓

$x'''' = \frac{7\pi}{4} + 2\pi = \frac{11\pi}{4}$ ✓

No others

$x = \left\{\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{5\pi}{2}, \frac{11\pi}{4}\right\}$

Question 9

Simplify the following expression:

$$\frac{\sin(2t)}{\cos(0-t)\cos(\frac{\pi}{2}+t)}$$

$$= \frac{2\sin t\cos t}{(\underset{1}{\cos 0}\cos t + \underset{0}{\sin 0}\sin t)(\underset{0}{\cos\frac{\pi}{2}}\cos t - \underset{1}{\sin\frac{\pi}{2}}\sin t)}$$

$$= \frac{2\sin t\cos t}{(\cos t)(-\sin t)}$$

$$= \frac{2}{-1} = -2$$

Question 10

Prove the following trigonometric identity:

$$\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta}=2\csc\theta$$

$$LHS = \frac{\sin\theta}{1-\cos\theta} + \frac{\sin\theta}{1+\cos\theta}$$

$$LCD = (1-\cos\theta)(1+\cos\theta)$$

$$= \frac{\sin\theta(1+\cos\theta) + \sin\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta + \sin\theta - \sin\theta\cos\theta}{1+\cos\theta-\cos\theta-\cos^2\theta}$$

$$= \frac{2\sin\theta}{1-\cos^2\theta} = \frac{2\sin\theta}{\sin^2\theta} = \frac{2}{\sin\theta} = 2\csc\theta = RHS$$

Question 11

Given $f(x) = 2\cos(2x+\pi) - \sqrt{3}$ defined over $x \in \left[0, \frac{3\pi}{2}\right]$:

$f(x) = 2\cos 2(x+\frac{\pi}{2}) - \sqrt{3}$ h

a) What is the Phase Shift (D) of f? $h = -\frac{\pi}{2}$

b) What is the Period (P) of f? $P = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$

c) What is the Range of f? $\min = k - a = -\sqrt{3} - 2 = -3.732$

$Ran f = [-3.732, 0.268]$ $Max = k + a = -\sqrt{3} + 2 = 0.268$

d) What are the zeros of f? $\rightarrow f(x) = 0$

$0 = 2\cos 2(x+\frac{\pi}{2}) - \sqrt{3}$

$\frac{\sqrt{3}}{2} = \frac{2\cos 2(x+\frac{\pi}{2})}{2}$

$\cos 2(x+\frac{\pi}{2}) = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ or $\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$\frac{2(x+\frac{\pi}{2})}{2} = \frac{\frac{\pi}{6}}{2}$

$x + \frac{\pi}{2} = \frac{\pi}{12}$

$x = \frac{\pi}{12} - \frac{\pi}{2} = -\frac{5\pi}{12}$

$x' = \frac{-5\pi}{12} + \pi = \frac{7\pi}{12}$ ✓

$x'' = \frac{7\pi}{12} + \pi = \frac{19\pi}{12}$

OR $\frac{2(x+\frac{\pi}{2})}{2} = \frac{\frac{11\pi}{6}}{2}$

$x + \frac{\pi}{2} = \frac{11\pi}{12}$

$x = \frac{11\pi}{12} - \frac{\pi}{2} = \frac{5\pi}{12}$ ✓

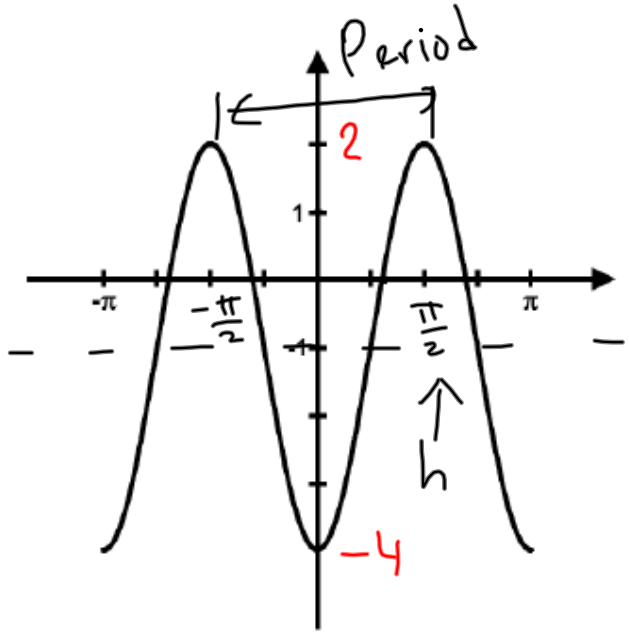
$x' = \frac{5\pi}{12} + \pi = \frac{17\pi}{12}$ ✓

Noothers

Zeros: $\left\{\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}\right\}$

Question 12

Determine the equation of the following sinusoidal function. Note that the amplitude parameter is positive.



$$a = \frac{\text{Max} - \text{min}}{2} = \frac{2-(-4)}{2} = 3$$

$$K = \frac{\text{Max} + \text{min}}{2} = \frac{2+(-4)}{2} = -1$$

$$\text{Period} = M_2 - M_1 = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$b = \frac{2\pi}{P} = \frac{2\pi}{\pi} = 2$$

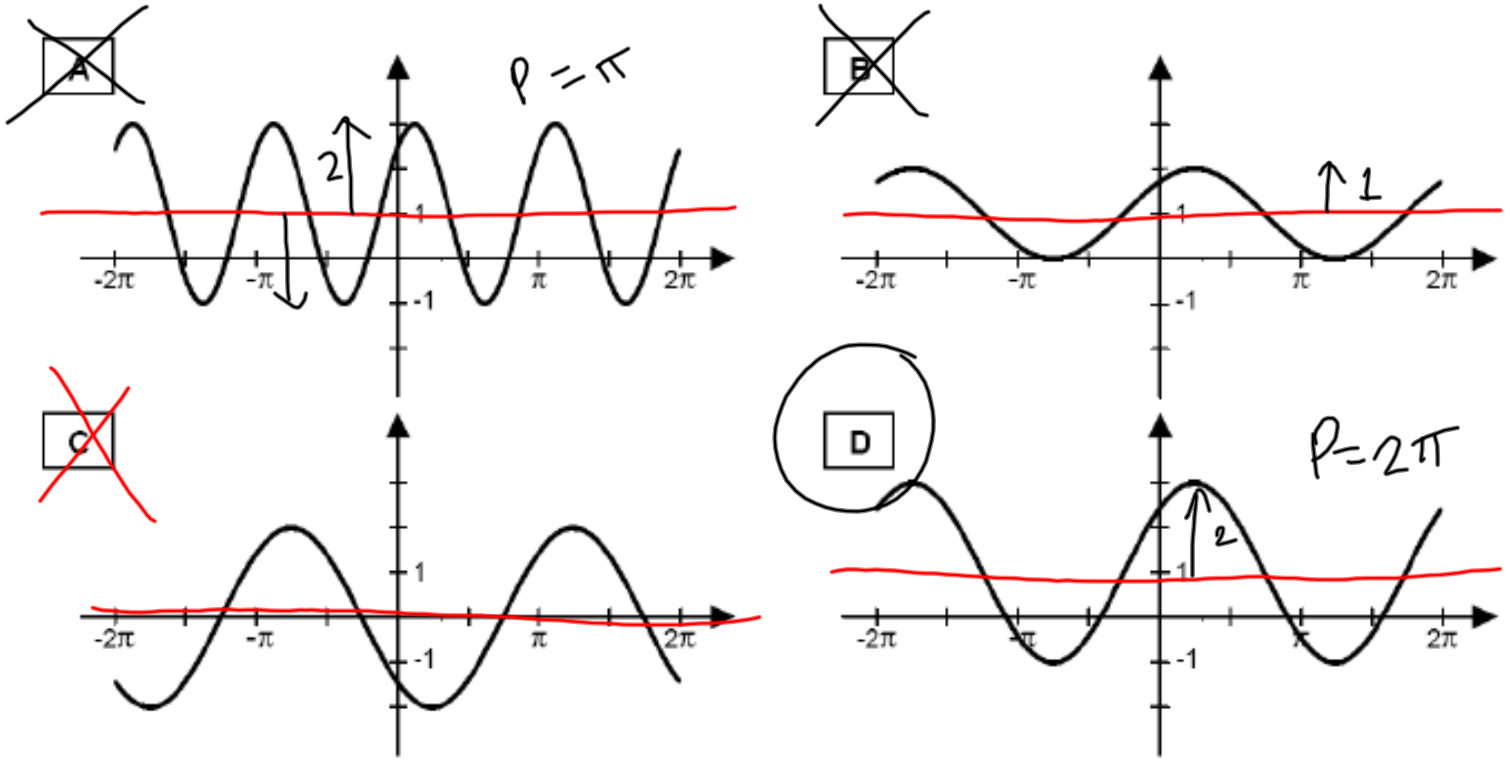
$$f(x) = a\cos b(x-h) + k$$

$$f(x) = 3\cos 2\left(x-\frac{\pi}{2}\right) - 1$$

Question 13

Among the following graphs, determine which corresponds to a function of the type $f(x)=A\cos(bx-h)+k$ where:

- The Amplitude is 2 units
- The vertical shift is +1 units
- The Period is 2π units



Question 14

$\frac{\pi}{2} \div 80\pi$

The position of a piston in a cylinder of an engine is determined by the following sinusoidal function:

$$p(t) = 3\sin(80\pi t + \frac{\pi}{2}) + 3 \longrightarrow 3 \sin 80\pi\left(t + \frac{1}{160}\right) + 3$$

Where $p(t)$ is the position after t seconds.

If the movement of the piston is modified such that the period is doubled without modifying the phase shift;

$$P = \frac{2\pi}{b} = \frac{2\pi}{80\pi} = \frac{1}{40}$$

a) What is the new equation of the position of the piston?
b) How many cycles will it complete in one minute?
c) To the nearest tenth, what is its position after 10 seconds?

a) $p'(t) = 3 \sin 40\pi\left(t + \frac{1}{160}\right) + 3$

$$P' = 2P = \frac{2}{40} = \frac{1}{20}$$

b) $P' = \frac{1}{20} \rightarrow f' = \frac{1}{P'} = \frac{1}{\frac{1}{20}} = 20$ cycles per second

$\times 60$

1200 cycles per minute

$$b' = \frac{2\pi}{P'} = \frac{2\pi}{\frac{1}{20}} = 40\pi$$

c) $p'(10) = 3\sin 40\pi\left(10 + \frac{1}{160}\right) + 3$

$= 3\sin \frac{40\pi}{160}\left(\frac{1601}{160}\right) + 3$

$= 3\sin \frac{1601\pi}{4} + 3 = 3\sin \frac{\pi}{4} + 3 = \frac{3\sqrt{2}}{2} + 3 = 5.1$

Question 15

Over a period of two years, oceanographers have compiled data on the mass of humpback whales. They observed that their weight varies according to the sinusoidal function:

$$m(t) = 25\sin\left(\frac{\pi t}{12}\right) + 80, \text{ where } t \in [0,24]$$

Where *m(t)* represents the mass of the whale in tons after *t* months.

a) What are the minimum and maximum masses of the whales over the period of observation?

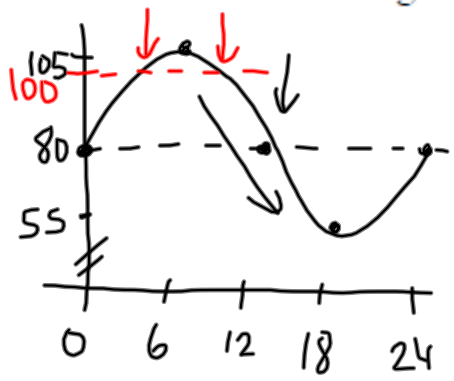
Min = k - a = 80 - 25 = 55 tons

Max = k + a = 80 + 25 = 105 tons

b) During the 12th month of observation, is the weight of the whales increasing or decreasing?

$b = \frac{\pi}{12} \rightarrow P = \frac{2\pi}{\frac{\pi}{12}} = 24$

Decreasing



c) During the entire observation period, at what times were the whales' masses exactly 100 tons? Round your answer to the nearest one-hundredth.

$m(t) = 25 \sin \frac{\pi}{12} t + 80$

$100 = 25 \sin \frac{\pi}{12} t + 80$

$\frac{20}{25} = \frac{25 \sin \frac{\pi}{12} t}{25}$

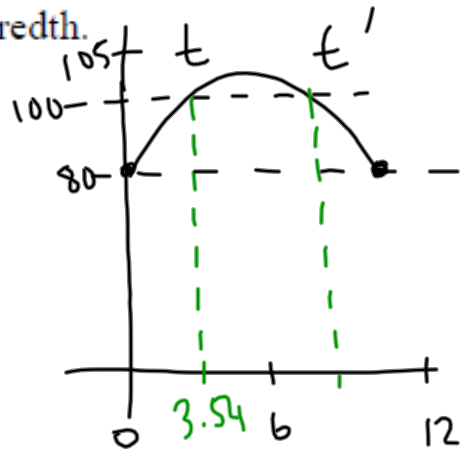
$0.8 = \sin \frac{\pi}{12} t$

$\sin^{-1}(0.8) = \frac{\pi}{12} t$

$\frac{0.9273}{\frac{\pi}{12}} = \frac{\frac{\pi}{12} t}{\frac{\pi}{12}}$

$t = 3.5420$

$= 3.54$ months



$t' = 12 - 3.54$

$= 8.46$ months